A combinatorial property of generic immersions of curves

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A *divide* (called "partage" in [AC2]) with r branches is the image P of a generic relative immersion of the union of $r, r \in \mathbf{N}$, copies of the unit interval $[0, 1]$ in the unit disk D^2. A *region* of a divide P is a connected component A of $D^2 \setminus P$, that does not meet the boundary ∂D^2. A *segment* of P is a connected component of the complement of the double points of P in P. A *sector* of P is a connected component of the germ at a double point of a region. We say that a divide is *cellular*, if P is connected and the closure in D^2 of each region of P is contractible. The *link* $L(P) \subset S^3$ of a divide P is obtained by a hodographic construction [AC4-5]:

$$L(P) := \{(x, u) \in TD^2 \mid x \in P, u \in T_x P, ||x||^2 + ||u||^2 = 1\}$$

The link $L(P)$ of a divide has a natural orientation. The complement $S^3 \setminus L(P)$ of the link of a connected divide P admits a fibration over the circle S^1, whose restriction to each oriented meridian of each component is a degree 1 map [AC5]. The fiber F_P is a surface of genus $\delta(P)$ with r boundary components, where $\delta(P)$ is the number of double points of P. Let $T_P : F_P \to F_P$ be the monodromy diffeomorphism of this fibration. A divide P is *simple* if P is connected and has at least one double point, such that there does not exist a relatively embedded copy of $[0, 1]$ in D^2, that cuts P transversally in one point and separates the double points of P in two non-empty sets.

Theorem 1 *The Lefschetz number of the monodromy T_P of a simple, cellular divide is 0.*

If the cellular divide P is the saddle level of a local real maximal deformation of a real equation of a plane curve singularity [S1-2,AC2,AC4,G-Z],

the Lefschetz number of the monodromy T_P equals the Lefschetz number of the monodromy of the singularity, and therefore equals 0 by [AC1]. The divide of Fig. 1 is obtained by a maximal real deformation from the singularity $x((y^2 - x^3)^2 - 4yx^4 - x^7)$ [G-Z]. So the Lefschetz number of its monodromy equals 0 although the divide is not cellular.
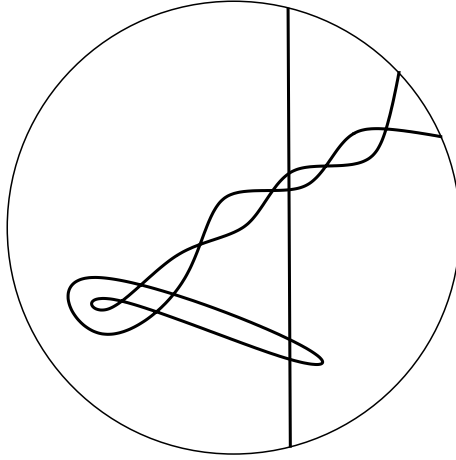

Fig. 1. Divide for the singularity $x((y^2 - x^3)^2 - 4yx^4 - x^7)$.

The Lefschetz number of the non-cellular divide of Fig. 2a equals 2. The Lefschetz number of the non-simple divide of Fig. 2b equals -1.
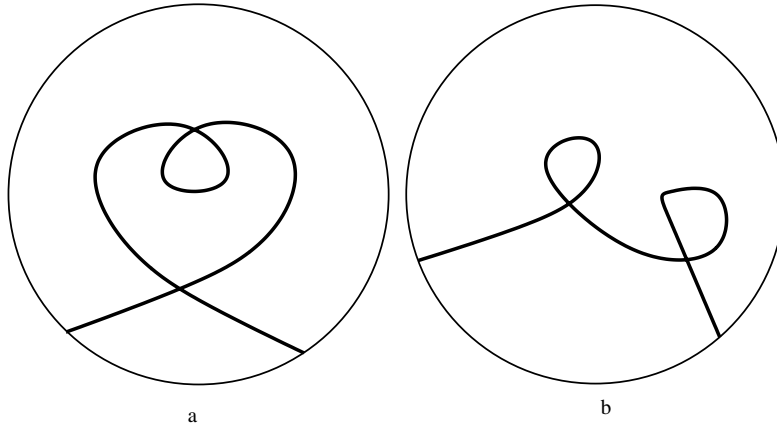

a   b

Fig. 2. a: non-cellular, b: non-simple.

Before giving the proof, we will recall some material about the Seifert form of the link of a connected divide, see [AC2,AC4, G-Z]. The connected

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components of $D^2 \setminus P$ will be signed with $\pm$, such that two components have opposite signs if their closures have a segment of P in common. Moreover, we choose a basepoint b_A in each region A. The *geometric Dynkin diagram* Γ of the divide P is the planar graph in D^2 obtained as follows. The vertices of Γ are the double points of P and the basepoints of the regions. The edges of Γ are arcs of two species. For each sector at a double point of P we draw an arc, that connects the double point with the basepoint of the region to which the sector belongs. Moreover, for each segment of P, we connect the basepoints of those regions, whose closures meet along the given segment by an arc having one transversal intersection with the segment. Pairs of vertices of Γ can be connected by several edges precisely if the divide is non-cellular.

The geometric Dynkin diagram was introduced by Sabir Gusein-Zade [G-Z] as system of gradient lines of a morse function on D^2. We number the vertices of Γ with $1 \leq i \leq \mu_P$ by taking first the basepoints of the $-$ regions, then the double points of P and finally the basepoints of the $+$ regions. The geometric Dynkin diagram defines the Seifert form S of the fibered link $L(P)$. On the vector space spanned by the set of vertices V of Γ we define an upper triangular nilpotent endomorphism $N := (n_{ij})$ by setting the matrix coefficient n_{ij} for $i < j$ equal to the number of edges of Γ that connect i and j.

From the tricollaring of Γ by $\{-, ., +\}$ one can conclude for any divide a basic fact, namely that the equality $N^3 = 0$ holds. A divide is a slalom divide of a rooted tree or of a disk-wide-web if and only if the equality $N^2 = 0$ holds, see [AC3,AC6].

The Seifert form is represented by $Id + N$. The monodromy action T_* on the first homology of F_P is represented by the matrix $^t(Id + N)^{-1} \circ (Id + N)$. For the Lefschetz number Λ_P of T this yields the expression

$$1 - \text{Tr}(Id -^t N + N -^t NN +^t N^2 N) = 1 - \mu_P + \text{Tr}(^t NN) - \text{Tr}(^t N^2 N)$$

For instance for a slalom divide we have

$$\Lambda = 1 - \mu_P + \text{Tr}(^t NN) = 1 - \mu_P + (\mu_P - 1) = 0.$$

Proof of theorem 1: For a cellular divide we have $n_{ij}^2 = n_{ij}$, hence

$$\text{Tr}(^t NN) = \sum_{ij} n_{ij}^2 = \sum_{ij} n_{ij} = e_P$$

where e_P is the number of edges of the geometric Dynkin diagram Γ. The term $\text{Tr}(^t N^2 N)$ has for a cellular divide also a combinatorial interpretation. We call

flag of the divide P a pair (a, b) of edges of Γ such that the edge a connects a basepoint of a $-$ region with a double point of P, while the edge b connects this double point with the basepoint of a $+$ region. Calling the number of flags f_P we have for a cellular divide $\mathrm{Tr}(^t N^2 N) = f_P$ and it remains to prove: $\mu_P - e_P + f_P = 1$.
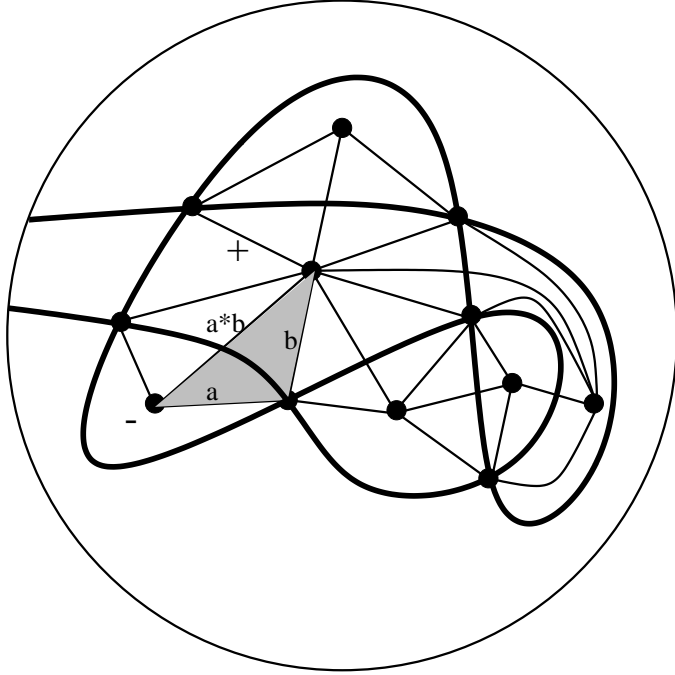

Fig. 3. Body of a divide with its partial triangulation.

We call *body* B_P of the divide P the union of the set of its double points with the closure of the union of its regions. For a simple divide P the Euler-Poincaré characteristic $\chi(B_P)$ equals 1. Each flag (a, b) of P defines a triangle $\Delta_{(a,b)} = (a, b, a * b) \subset B_P$ in the the body, where $a * b$ is the edge of Γ which connects the non common endpoints of a and b. Let $S \subset B_P$ be the union of the set of double points of P, of the edges of Γ and of the triangles associated to the flags. This system of vertices, edges and triangles is a triangulation of S. The body B_P collapses onto S, see Fig 3. Hence we conclude

$$\mu_P - e_P + f_P = \chi(S) = \chi(B_P) = 1,$$

which completes the proof.

I like to thank Marc Burger for his remark, that $\mu_P - e_P + f_P$ should be interpreted as Euler-Poincaré characteristic.

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Remark: The property $N^3 = 0$ that, as we have seen here above, holds for any divide, indicates that the computation of the traces of the iterates of the monodromy is related to the random walk on the geometric Dynkin diagram Γ. We ask for an expression of $\text{Tr}(T^k)$ in terms of the generating function of the random walk on the Dynkin diagram Γ, see [D-J].

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